Symmetric radial decreasing rearrangement can increase the fractional Gagliardo norm in domains

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Abstract

We show that the symmetric radial decreasing rearrangement can increase the fractional Gagliardo semi-norm in domains.

1 Introduction

For any Borel set A in $\mathbb{R}^n$ with $|A| < \infty$ ($|A|$ denotes the Lebesgue measure of A), define A^*, the symmetric rearrangement of A as the open ball

$$A^* = \{x : |x| < (|A|/\alpha_n)^{\frac{1}{n}}\},$$

where $\alpha_n = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)$ is the volume of the unit ball. If $|A| = 0$, then $A^* = \varnothing$ and for later purposes we conveniently define $\chi_\varnothing \equiv 0$. Denote by $\mathscr{U}_0$ the space of Borel measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mu_u(t) = |\{x : |u(x)| > t\}| \text{ is finite for all } t > 0.$$

Observe that $\mu_u(\cdot)$ is right-continuous, non-increasing and (by the Lebesgue dominated convergence theorem) $\lim_{t\to\infty} \mu_u(t) = 0$. For any $u \in \mathscr{U}_0$, define the symmetric decreasing rearrangement u^* as

$$u^*(x) = \int_0^\infty \chi_{\{|u|>t\}^*}(x)dt = \sup\{t : |\{|u| > t\}| > \alpha_n |x|^n\}.$$

Since μ_u decays to zero as $t \to \infty$, we have $0 \le u^*(x) < \infty$ for any $x \neq 0$, whereas $u^*(0)$ may be ∞. Evidently, the function u^* is radial, non-increasing in $|x|$, and satisfy

$$\{|u| > t\}^* = \{u^* > t\}, \quad \forall\, t > 0.$$

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From this one can deduce $|\{|u| > t\}| = |\{u^* > t\}|, \forall\, t > 0$ and $\|u^*\|_p = \|u\|_p$ for all $1 \le p \le \infty$. Note that it follows from the level set characterisation that any uniform translation of u does not change u^*, namely if for any $x_0 \in \mathbb{R}^n$, we define $u_{x_0}(x) = u(x - x_0)$, then

$$(u_{x_0})^* = u^*. \tag{1}$$

This simple property will be used without explicit mentioning later. On the other hand, the effect of rearrangement on the gradient of the function is more complex and interesting. Let u be a nonnegative smooth function that vanishes at infinity. The Pólya-Szegö [10] inequality states that for $1 \le p < \infty$,

$$\int_{\mathbb{R}^n} |\nabla u|^p \ge \int_{\mathbb{R}^n} |\nabla u^*|^p.$$

Brothers-Ziemer [2] gave a characterization of the equality case under the assumption that the distribution function of u is absolutely continuous. This Pólya-Szegö inequality also holds for every bounded open set $\Omega \subset \mathbb{R}^n$. That is, for every nonnegative $u \in C_c^\infty(\Omega)$, we also have

$$\int_{\Omega} |\nabla u|^p \ge \int_{\Omega^*} |\nabla u^*|^p.$$

As a matter of fact, one can show that for every $u \in W_0^{1,p}(\Omega)$, one has $u^* \in W_0^{1,p}(\Omega^*)$ and the above inequality holds.

We are interested in the effect of symmetric decreasing rearrangement for fractional Sobolev inequalities. For $0 < \sigma < 1$ and $1 \le p < \infty$, we define the space $\mathring{W}^{\sigma,p}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_{\mathring{W}^{\sigma,p}(\Omega)} = \left(\iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\sigma p}}\, \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{p}}.$$

It was shown in Theorem 9.2 in Almgren-Lieb [1] that

$$\|u\|_{\mathring{W}^{\sigma,p}(\mathbb{R}^n)} \ge \|u^*\|_{\mathring{W}^{\sigma,p}(\mathbb{R}^n)}.$$

Characterizations of the equality case have been given in Burchard-Hajaiej [3] and Frank-Seiringer [7]. Motivated by the Pólya-Szegö inequality in domains, we would like to investigate whether the above inequality holds for bounded open sets Ω. That is, do we have

$$\|u\|_{\mathring{W}^{\sigma,p}(\Omega)} \ge \|u^*\|_{\mathring{W}^{\sigma,p}(\Omega^*)}? \tag{2}$$

Another motivation of the above question comes from Frank-Jin-Xiong [6], where the authors study the best constants of fractional Sobolev inequalities on domains. A classical result of Lieb [8] implies that

$$S(n, \sigma, \mathbb{R}^n) \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2\sigma}}\, \mathrm{d}x \right)^{\frac{n-2\sigma}{n}} \le \|u\|^2_{\mathring{W}^{\sigma,2}(\mathbb{R}^n)} \quad \text{for all } u \in \mathring{W}^{\sigma,2}(\mathbb{R}^n), \tag{3}$$

where $S(n, \sigma, \mathbb{R}^n) = \frac{2^{1-2\sigma}\omega_n^{\frac{2\sigma}{n}}\pi^{\frac{n}{2}}\Gamma(2-\sigma)}{\sigma(1-\sigma)\Gamma(\frac{n-2\sigma}{2})}$ and ω_n is the volume of the unit $n-$dimensional sphere. Moreover, the equality in (3) holds if and only if $u(x) = (1 + |x|^2)^{-\frac{n-2\sigma}{2}}$ up to translating and scaling. These follow from the fact that the sharp fractional Sobolev inequality is a dual inequality of the sharp Hardy-Littlewood-Sobolev inequality. For an open set $\Omega \neq \mathbb{R}^n$, if $\sigma \in (1/2, 1)$ and $n \geq 2$, then there exists a positive constant $\underline{S}(n, \sigma)$ depending only on n, σ but *not* on Ω such that

$$\underline{S}(n,\sigma) \left(\int_\Omega |u|^{\frac{2n}{n-2\sigma}} \,\mathrm{d}x \right)^{\frac{n-2\sigma}{n}} \leq \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\mathrm{d}y \quad \text{for all } u \in \mathring{W}^{\sigma,2}(\Omega). \tag{4}$$

This inequality is called the fractional Sobolev inequality in domain Ω. It is included in Theorem 1.1 in Dyda-Frank [5]. It actually follows from (3) and a fractional Hardy inequality of Dyda [4], Loss-Sloane [9] and Dyda-Frank [5] (by using similar arguments to the proof of Theorem 1.2 here; see the remark in the end of this paper).

In Frank-Jin-Xiong [6], they studied the best constant in (4):

$$S(n,\sigma,\Omega) := \inf \left\{ \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\mathrm{d}y \mid u \in C_c^\infty(\Omega), \int_\Omega |u|^{\frac{2n}{n-2\sigma}} \,\mathrm{d}x = 1 \right\}.$$

It was proved in [6] that this best constant $S(n, \sigma, \Omega)$ actually depends on the domain Ω, and can be achieved in many cases such as in the half spaces $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n, x_n > 0\}$ or some smooth bounded domains, which is in contrast to the classical Sobolev inequalities in domains. Let B_r be the ball of radius r centered at the origin, and $B_1^+ = B_1 \cap \mathbb{R}^n_+$. Suppose $\sigma \in (1/2, 1)$, and Ω is a C^2 bounded open set such that $B_1^+ \subset \Omega \subset \mathbb{R}^n_+$, then it was proved in [6] that both $S(n, \sigma, \mathbb{R}^n_+)$ and $S(n, \sigma, \Omega)$ are achieved, and there holds the inequality

$$S(n,\sigma,\Omega) < S(n,\sigma,\mathbb{R}^n_+) < S(n,\sigma,\mathbb{R}^n).$$

On the other hand, from (4), we have that for $\sigma \in (1/2, 1)$, $S(n, \sigma, \Omega) \geq \underline{S}(n, \sigma) > 0$ for every open set Ω. An interesting question left open is to find the value of $\inf_\Omega S(n, \sigma, \Omega)$ for $\sigma \in (1/2, 1)$, where the infimum is taken over all bounded open sets Ω. A conjecture is that $\inf_\Omega S(n, \sigma, \Omega)$ is achieved by a ball, which could follow from (2) . However, we show in this paper that (2) is false.

Theorem 1.1. *Let $n \geq 1$ and Ω be any nonempty open set in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $\sigma \in (0, 1)$ and $p \in (0, \infty)$. There exists a nonnegative $u \in C_c^\infty(\Omega)$ such that*

$$\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y < \iint_{\Omega^*\times\Omega^*} \frac{|u^*(x)-u^*(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y.$$

We will prove this theorem in the next section by using an explicit computation.

Remark. *Theorem 1.1 holds in particular when Ω is an open ball centered at the origin (so that $\Omega^* = \Omega$).*

Remark. *In [1] (see Corollary 2.3 therein), a general rearrangement inequality is shown to hold for convex integrands. Namely, if $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is convex with $\Psi(0) = 0$, then for every nonnegative $L^1(\mathbb{R}^n)$ function W, every nonnegative f, $g \in \mathscr{U}_0$ with $\Psi \circ f$, $\Psi \circ g \in L^1(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(|f(x)-g(y)|) W(x-y) dxdy \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(|f^*(x)-g^*(y)|) W^*(x-y) dxdy.$$

Our Theorem 1.1 shows that such a general result cannot hold if $\mathbb{R}^n$ is replaced by a domain Ω of finite measure on the left-hand side (and correspondingly by Ω^* on the right-hand side).

On the other hand, we have the following estimate.

Theorem 1.2. *Let $n \geq 1$, $\sigma \in (0,1)$ and $p \in (0,\infty)$ be such that $\sigma p > 1$. Then there exists a positive constant C depending only on n, σ and p such that*

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u^*(x) - u^*(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y \leq C \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y$$

for all open sets $\Omega \subset \mathbb{R}^n$ and all nonnegative $u \in C_c^\infty(\Omega)$. In particular,

$$\iint_{\Omega^* \times \Omega^*} \frac{|u^*(x) - u^*(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y \leq C \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y.$$

2 Proofs

We begin with the following simple lemma. Recall that for any two sets A and B, their symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Remark. *For an open set $\Omega \subset \mathbb{R}^n$, $|\Omega^* \triangle \Omega| = 0$ if and only if $\Omega = \Omega^*$.*

Lemma 2.1. *Let Ω be an open and bounded set in $\mathbb{R}^n$.*

(i). Suppose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is radial and strictly decreasing, i.e. $f(x) > f(y)$ if $|x| < |y|$. Then

$$\int_{\Omega^*} f(x)\,\mathrm{d}x > \int_{\Omega} f(x)\,\mathrm{d}x \quad \text{if } |\Omega^* \triangle \Omega| > 0. \tag{5}$$

(ii). Suppose $\overline{B}_\delta \subset \Omega$ for some $\delta > 0$, and let $f \in L^1(\mathbb{R}^n \setminus \overline{B}_\delta)$ be radial and strictly decreasing. Then

$$\int_{\mathbb{R}^n \setminus \Omega} f(x)\,\mathrm{d}x > \int_{\mathbb{R}^n \setminus \Omega^*} f(x)\,\mathrm{d}x \quad \text{if } |\Omega^* \triangle \Omega| > 0. \tag{6}$$

Remark. *The main example is $f(x) = |x|^{-\alpha}$ for some $\alpha > 0$. Similar proof as below can show the well-known inequality that for any Borel measure $A \subset \mathbb{R}^n$ with $|A| < \infty$, $x_0 \in \mathbb{R}^n$, and $\delta > 0$, one has*

$$\int_{\mathbb{R}^n \setminus A} |x - x_0|^{-n-\delta} dx \geq \int_{\mathbb{R}^n \setminus A^*} |x|^{-n-\delta} dx = \mathrm{const} \cdot |A|^{-\frac{\delta}{n}}.$$

This inequality can be used to establish fractional Sobolev embedding. We should stress that in our case one needs strict inequality and for this reason we impose strict monotonicity on f.

Proof. Let r be the radius of Ω^*.

We prove (i) first. Notice that

$$\int_{\Omega^*} f(x)\,\mathrm{d}x - \int_{\Omega} f(x)\,\mathrm{d}x = \int_{\Omega^* \setminus \Omega} f(x)\,\mathrm{d}x - \int_{\Omega \setminus \Omega^*} f(x)\,\mathrm{d}x.$$

Since $|\Omega^*| = |\Omega|$, we have $|\Omega \setminus \Omega^*| = |\Omega^* \setminus \Omega| = \frac{1}{2}|\Omega^* \triangle \Omega| > 0$. Since f is radial and strictly decreasing, we have

$$\int_{\Omega^* \setminus \Omega} f(x)\,\mathrm{d}x > f(r)|\Omega^* \triangle \Omega|,$$
$$\int_{\Omega \setminus \Omega^*} f(x)\,\mathrm{d}x < f(r)|\Omega^* \triangle \Omega|.$$

Hence, the inequality (5) follows.

To prove (ii), we notice $\overline{B}_\delta \subset \Omega^*$ by the assumption, and

$$\int_{\mathbb{R}^n \setminus \Omega} f(x)\,\mathrm{d}x - \int_{\mathbb{R}^n \setminus \Omega^*} f(x)\,\mathrm{d}x = \int_{\Omega^* \setminus \Omega} f(x)\,\mathrm{d}x - \int_{\Omega \setminus \Omega^*} f(x)\,\mathrm{d}x.$$

Hence, the inequality (6) follows the same as above. □

Proof of Theorem 1.1. Our proof of the general case in Theorem 1.1 is inspired by that of the special case Ω being a ball. So we will provide the proof of Theorem 1.1 for $\Omega = B_1$ first.

Let $\eta \in C_c^\infty(B_1)$ be a radially decreasing function such that $\eta(x) = 1$ for $|x| \le 1/2$. Let $\varepsilon \in (0, 1/2)$ which will be chosen very small,

$$x_\varepsilon = (1 - \varepsilon, 0, \cdots, 0),$$

and

$$u_\varepsilon = \eta\left(\frac{x - x_\varepsilon}{\varepsilon}\right).$$

Since we assumed that η is smooth, nonnegative, and radially decreasing, it is clear that

$$u_\varepsilon^* = \eta\left(\frac{x}{\varepsilon}\right).$$

Therefore,

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_\varepsilon^*(x) - u_\varepsilon^*(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y. \tag{7}$$

Since $B_1^* = B_1$ and

$$\begin{aligned}&\iint_{B_1 \times B_1} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y\\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y - 2\int_{B_1} u_\varepsilon^p(x)\left(\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\right)\mathrm{d}x,\end{aligned}$$

we only need to show that

$$\int_{B_1} u_\varepsilon^p(x)\left(\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\right)\mathrm{d}x > \int_{B_1} (u_\varepsilon^*(x))^p\left(\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\right)\mathrm{d}x. \tag{8}$$

First, since u^* is supported in B_ε, we have

$$\begin{aligned}\int_{B_1} (u_\varepsilon^*(x))^p \left(\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|x-y|^{n+\sigma p}} \,\mathrm{d}y \right) \mathrm{d}x &= \int_{B_\varepsilon} (u_\varepsilon^*(x))^p \left(\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|x-y|^{n+\sigma p}} \,\mathrm{d}y \right) \mathrm{d}x \\ &\le C \int_{B_\varepsilon} (u_\varepsilon^*(x))^p \,\mathrm{d}x = C\varepsilon^n \int_{B_1} \eta^p(x)\,\mathrm{d}x,\end{aligned}$$

where (as well as in the below) C is a positive constant independent of ε.

Secondly, since u is supported in $B_\varepsilon(x_\varepsilon)$, we have

$$\begin{aligned}\int_{B_1} u_\varepsilon^p(x) \left(\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|x-y|^{n+\sigma p}} \,\mathrm{d}y \right) \mathrm{d}x &\ge \int_{B_\varepsilon(x_\varepsilon)} u_\varepsilon^p(x) \left(\int_{\{y:\, y_1 \ge 1\}} \frac{1}{|x-y|^{n+\sigma p}} \,\mathrm{d}y \right) \mathrm{d}x \\ &\ge C \int_{B_\varepsilon(x_\varepsilon)} \frac{u_\varepsilon^p(x)}{(1-x_1)^{\sigma p}} \,\mathrm{d}x \\ &\ge C\varepsilon^{-\sigma p} \int_{B_\varepsilon(x_\varepsilon)} u_\varepsilon^p(x)\,\mathrm{d}x = C\varepsilon^{n-\sigma p} \int_{B_1} \eta^p(x)\,\mathrm{d}x,\end{aligned}$$

where in the second inequality, we used that for $x = (x_1, \cdots, x_n)$ with $x_1 < 1$,

$$\int_{\{y:\, y_1 \ge 1\}} \frac{1}{|x-y|^{n+\sigma p}} \,\mathrm{d}y = C(1-x_1)^{-\sigma p}.$$

This proves (8), and thus Theorem 1.1 for $\Omega = B_1$, if we choose ε sufficiently small.

Now let us consider the general case where Ω is not a ball. Since Ω is an open set and the Gagliardo semi-norm is translation invariant and dilation invariant (and also by (1)), without loss of generality, we may assume Ω contains B_1. Again, let $\eta \in C_c^\infty(B_1)$ be a radially decreasing function such that $\eta(x) = 1$ for $|x| \le 1/2$. Define for $\varepsilon \in (0,1)$,

$$u_\varepsilon(x) = \eta\left(\frac{x}{\varepsilon}\right).$$

Hence,

$$u_\varepsilon^*(x) = \eta\left(\frac{x}{\varepsilon}\right),$$

and thus, (7) also holds. Since

$$\begin{aligned}&\iint_{\Omega\times\Omega} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\mathrm{d}y - 2\int_\Omega u_\varepsilon^p(x) \left(\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+\sigma p}} \,\mathrm{d}y \right) \mathrm{d}x,\end{aligned}$$

we only need to check the inequality

$$\int_\Omega u_\varepsilon^p(x) F(x) dx > \int_{\Omega^*} (u_\varepsilon^*(x))^p \tilde{F}(x) dx, \tag{9}$$

where

$$F(x)=\int_{\mathbb{R}^n\setminus\Omega}\frac{1}{|x-y|^{n+p\sigma}}dy\quad\text{and}\quad \tilde{F}(x)=\int_{\mathbb{R}^n\setminus\Omega^*}\frac{1}{|x-y|^{n+p\sigma}}dy. \tag{10}$$

Noticing the support of η, this reduces to checking the inequality

$$\int_{B_1}\eta^p(x)F(\varepsilon x)dx>\int_{B_1}\eta^p(x)\tilde{F}(\varepsilon x)dx. \tag{11}$$

Since Ω is an open and is not a ball, we have $|\Omega^*\setminus\Omega|=|\Omega\setminus\Omega^*|>0$. Then it follows from (6) in Lemma 2.1 that $F(0)>\tilde{F}(0)$. Hence, the inequality (11) holds for all ε sufficiently small by using the Lebesgue dominated convergence theorem. Theorem 1.1 is proved.

We remark that the above proof for the general case where Ω is not a ball can also be used to prove the case when $\Omega=B_1$, which is as follows. Let η be the same as before, $|\bar{x}|=1/2$ and define

$$u_\varepsilon(x)=\eta\left(\frac{x-\bar{x}}{\varepsilon}\right).$$

Hence,

$$u_\varepsilon^*(x)=\eta\left(\frac{x}{\varepsilon}\right).$$

As above, we only need to check the inequality (9). Since $\Omega=B_1$, we have $\Omega^*=\Omega$ and $F=\tilde{F}$. Thus, by change of variables and noticing the support of η, this reduces to checking the inequality

$$\int_{B_1}\eta^p(x)F(\bar{x}+\varepsilon x)dx>\int_{B_1}\eta^p(x)F(\varepsilon x)dx. \tag{12}$$

Since

$$F(\bar{x})=\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|\bar{x}-y|^{n+p\sigma}}dy=\int_{\mathbb{R}^n\setminus B_1(\bar{x})}\frac{1}{|z|^{n+p\sigma}}dz>\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|z|^{n+p\sigma}}dz=F(0),$$

where we used (6) in the last inequality (noticing $(B_1(\bar{x}))^*=B_1$), the inequality (12) holds for all ε sufficiently small by using the Lebesgue dominated convergence theorem. □

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. We only need to consider the case where $\Omega\subset\mathbb{R}^n$ is an open set that satisfies $|\mathbb{R}^n\setminus\Omega|>0$. Let $u\in C_c^\infty(\Omega)$ be a nonnegative function.

Then

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u^*(x)-u^*(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y\\
&\le\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y\\
&=\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y+2\int_\Omega u^p(x)\left(\int_{\mathbb{R}^n\setminus\Omega}\frac{1}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\right)\mathrm{d}x.
\end{aligned} \tag{13}$$

As in Loss-Sloane [9] and Dyda-Frank [5], we denote

$$d_\omega(x) = \inf\{|t| : x + t\omega \notin \Omega\}, \quad x \in \mathbb{R}^n, \quad \omega \in \mathbb{S}^{n-1},$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere, and

$$m_\alpha(x) = \left(\frac{2\pi^{\frac{n-1}{2}}\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})}\right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{S}^{n-1}} \frac{1}{d_\omega(x)^\alpha}\,\mathrm{d}\omega\right)^{-\frac{1}{\alpha}}.$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+\sigma p}}\,\mathrm{d}y \le \int_{\mathbb{S}^{n-1}} \mathrm{d}\omega \int_{d_\omega(x)}^\infty \frac{1}{r^{n+\sigma p}}\,\mathrm{d}r &= (n+\sigma p-1)\int_{\mathbb{S}^{n-1}} \frac{1}{d_\omega(x)^{\sigma p}}\mathrm{d}\omega \\ &= \frac{C(n,\sigma,p)}{(m_{\sigma p}(x))^{\sigma p}}\end{aligned}$$

for some constant $C(n,\sigma,p)$ depending only on n, σ and p, but *not* on Ω. Thus, we have

$$\begin{aligned}\int_\Omega u^p(x)\left(\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\right)\mathrm{d}x &\le C(n,\sigma,p)\int_\Omega \frac{u^p(x)}{(m_{\sigma p}(x))^{\sigma p}}\,\mathrm{d}x \\ &\le C(n,\sigma,p)\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y, \qquad (14)\end{aligned}$$

where we use Theorem 1.2 (fractional Hardy inequality) of Loss-Sloane [9] in the last inequality. Therefore, combining (13) and (14), we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u^*(x)-u^*(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y \le C(n,\sigma,p)\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y.$$

Theorem 1.2 is proved. □

Remark. *The above proof of Theorem 1.2 can be used to prove* (4). *Indeed, if $n \ge 2$, $\sigma \in (0,1)$ and $1 < \sigma p < n$, then for every open set $\Omega \ne \mathbb{R}^n$ and all $u \in C_c^\infty(\Omega)$, we have*

$$\begin{aligned}\left(\int_\Omega |u(x)|^{\frac{np}{n-\sigma p}}\,\mathrm{d}x\right)^{\frac{n-\sigma p}{n}} &= \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{np}{n-\sigma p}}\,\mathrm{d}x\right)^{\frac{n-\sigma p}{n}} \\ &\le C(n,\sigma,p)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y \\ &\le C(n,\sigma,p)\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where in the first inequality we used the classical fractional Sobolev inequality in $\mathbb{R}^n$, and in the second inequality we used (13) *and* (14).

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